THEORY OF INSTANTANEOUSLY RIGID NETS

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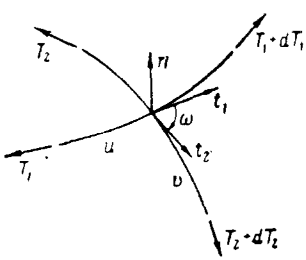
As used here, the term "net" denotes a system of flexible filaments directed along two one-parameter families of lines on a surface. Kinematic analysis of a net as a discrete rod system testifies to its multiple geometrical variability regardless of the conditions of attachment on the contour. However, the kinematic mobility of variable rod systems for some relationship among the geometrical parameters is limited to infinitesimal displacements. Such systems with specially selected geometrical parameters were first investigated by Rabinovich [1], who called them "instantaneously rigid" systems [2, 3]. With this approach, it is convenient to substitute the following static criterion for the kinematic criterion of an instantaneously rigid system formulated in [1] and stated above: a variable system that allows initial stresses is instantaneously rigid if its equilibrium is stable in the preliminary state of stress.

1. In order to derive the equilibrium conditions, let us consider an infinitely small element of the net in the neighborhood of one of its nodes (see accompanying figure). This element is bounded by the coordinate lines u and v which we direct along the two families of filaments; the triple of unit coordinate vectors $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{n}$ forms a movable trihedron on the surface. In the absence of an external stress, the equilibrium condition for the isolated net element is of the form



$$\frac{\partial}{\partial u}(\mathbf{T}_1 ds_2)\, du + \frac{\partial}{\partial v}(\mathbf{T}_2 ds_1)\, dv = 0 \qquad (1.1)$$

where

$$\mathbf{T}_1 = T_1\mathbf{t}_1, \quad \mathbf{T}_2 = T_2\mathbf{t}_2$$

are stress vectors in the filaments referred, respectively, to the unit increments of the linear elements

$$ds_2 = Bdv = 1, \quad ds_1 = Adu = 1$$

We denote the Lamé parameters by A and B. Condition (1.1) applies to the deformed state of the system, when the filaments can be assumed inextensible.

Carrying out the differentiation in (1.1) with the aid of Gauss' derivation formulas [4] and dividing through by $du\,dv$, we have (1.2)

$$T_{1,u}\frac{1}{A}\mathbf{t}_1 + T_1(\sigma_1\mathbf{n} - \varkappa_1\mathbf{b}_1) + T_1\mathbf{t}_1\frac{B_u}{AB} + T_{2,v}\frac{1}{B}\mathbf{t}_2 + T_2(\sigma_2\mathbf{n} - \varkappa_2\mathbf{b}_2) + T_2\mathbf{t}_2\frac{A_v}{AB} = 0$$

where σ and χ are the normal and geodetic curvatures of the filaments, and $\mathbf{b}$ is the unit vector along the binormal.

Equilibrium equation (1.2) has been obtained in a special coordinate system; from now on, however, it will be more convenient to make use of the invariant expression for this equation

$$\begin{aligned} &T_{1,i}u^i\mathbf{t}_1 + T_1(\sigma_1\mathbf{n} - \varkappa_1\mathbf{b}_1) + T_1\mathbf{t}_1A_iu^i + T_{2,i}v^i\mathbf{t}_2 + \\ &+ T_2(\sigma_2\mathbf{n} - \varkappa_2\mathbf{b}_2) + T_2\mathbf{t}_2A_iv^i = 0 \qquad (i = 1, 2) \end{aligned} \tag{1.3}$$

where u^i and v^i are the direction unit vectors of the net lines, and A_i is the so-called rhombic vector of the net which is determined unambiguously by any net [5].

The validity of notation (1.3) may be verified by substituting in the values assumed by the components of the vectors u^i, v^i and A_i in the special coordinate system,

$$u^i\left(\frac{1}{A},\ 0\right), \qquad v^i\left(0,\ \frac{1}{B}\right), \qquad A_i\left(-\frac{B_u}{B},\ -\frac{A_v}{A}\right) \tag{1.4}$$

Clearly, this substitution leads to condition (1.2); in keeping with the tensor character of equation (1.3), this guarantees the validity of the latter in an arbitrary coordinate system.

2. Scalar multiplication of equation (1.3) by $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{n}$ leads to a system of three equations that express the equilibrium conditions as projections on the corresponding axes,

$$\begin{aligned} &(T_{1,i} - T_1A_i)\,u^i \sin\omega - T_1\varkappa_1\cos\omega + T_2\varkappa_2 = 0 \\ &(T_{2,i} - T_2A_i)\,v^i \sin\omega + T_2\varkappa_2\cos\omega - T_1\varkappa_1 = 0 \\ &T_1\sigma_1 + T_2\sigma_2 = 0 \end{aligned} \tag{2.1}$$

where ω is the angle between the directional unit vectors of the net (the net angle).

Definition. A net for which the system of equilibrium equations (2.1) permits of a non-zero solution will be referred to as a static net.

From the third equation of system (2.1) we see that if the curvatures σ_1 and σ_2 are of the same sign, the stresses T_1 and T_2 are of different signs, and vice versa; the case where both curvatures are simultaneously equal to zero must be considered separately. But the equilibrium of the net is stable only if T_1 and T_2 are both tensile stresses. Hence, only those static nets which lie on surfaces of negative Gaussian curvature whose normals of filament curvature are of opposite sign at each point may be considered instantaneously rigid nets.

Before excluding the case $\sigma_1 = \sigma_2 = 0$, we note that this condition characterizes the asymptotic net of the surface, as well as an arbitrary plane net. The last equation of (2.1) is satisfied identically, while the first two form a canonical hyperbolic system whose characteristics coincide with the directions of the filaments, i.e. with the asymptotic lines on the surface. The Cauchy problem (specifically, the characteristic problem) for this system has a solution in the regular domain of the net which is completely determined once the stresses T_1 and T_2 are specified along some initial line (or along two intersecting characteristics, respectively). Since these stresses can both be made positive, the asymptotic net is an instantaneously rigid net at least locally.

3. We introduce the resolving function T of system (2.1) in terms of the formulas

$$T_1 = T\sigma_2, \qquad T_2 = T\sigma_1 \tag{3.1}$$

The third equation is then satisfied identically, and the first two can be combined into a single vector equation

$$\varphi_k \equiv \partial_k \ln T = A_k - \frac{\cos\omega}{\sin^2\omega}(\varkappa_1 v_k + \varkappa_2 u_k) - \tag{3.2}$$

$$- \frac{1}{\sin\omega}\left(\frac{\sigma_{1,i}}{\sigma_1} v^i u_k - \frac{\sigma_{2,i}}{\sigma_2} u^i v_k\right) + \frac{1}{\sin^2\omega}\left(\frac{\sigma_1}{\sigma_2}\varkappa_2 v_k + \frac{\sigma_2}{\sigma_1}\varkappa_1 u_k\right) \qquad \left(\partial_k = \frac{\partial}{\partial u^k}\right)$$

by minor manipulations.

The discriminant bivector $\varepsilon_{ik} = u_i v_k - u_k v_i$ is used here as a vector for permuting the subscripts [5].

The structure of the expression on the right-hand side of equation (3.2) enables us to say that it represents some vector φ_k belonging to the surface; this vector is determined unambiguously by the given net.

Definition. The vector φ_k whose components are given by the right-hand side of equation (3.2) is called the static vector of the net.

Theorem 3.1. In order for a net to be static, it is necessary and sufficient that its static vector be gradient.

In fact, since the gradient vector $\delta_k \ln T$ appears on the left-hand side of equation (3.2), it is quite clear that this equation, as well as system (2.1) permits of a non-trivial solution if and only if the vector φ_k is also gradient, i.e. if the condition

$$\text{curl}\ \varphi = 0 \tag{3.3}$$

is satisfied.

This is the necessary and sufficient condition for the existence of a function φ such that $\varphi_k = \partial_k\varphi$. We call the function ϕ the static potential of the net; by (3.2) the quantity required

$$T = Ce^{\varphi} \tag{3.4}$$

The invariant criterion of an instantaneously rigid net therefore consists of condition (3.3) imposed on the static vector in combination with the foregoing condition as regards the opposite signs of the normal curvatures.

If a given net is taken as the coordinate net, then regardless of the parametrization established thereon, the characteristic condition (3.3) assumes the form

$$\frac{\partial}{\partial u}\left(\frac{A}{B}\Gamma_{22}^1 \cos\omega + \frac{N}{L}\Gamma_{11}^2\right) - \frac{\partial}{\partial v}\left(\frac{B}{A}\Gamma_{11}^2 \cos\omega + \frac{L}{N}\Gamma_{22}^1\right) = \\ = \frac{\partial^2}{\partial u\, \partial v}\ln\left(\frac{B}{A}\frac{L}{N}\right) \tag{3.5}$$

where Γ_{ij}^k are Christoffel symbols, and L and N are the extreme coefficients of the second quadratic form of the surface.

4. The problem of the existence and number of static nets on any surface may be formulated as follows: Is it possible to choose a second one-parameter family of lines for a given one-parameter family in such a way as to obtain a static net?

Let u^i be the direction unit vector of the first family; we will consider the unit vector v^i of the second family as the unit vector u^i rotated by some angle ω in such a way that

$$v^i = u^i \cos\omega + u'^i \sin\omega \tag{4.1}$$

where u'^i is the complement of the vector u^i.

The geodetic and normal curvatures of the second-family lines are expressed in terms of the invariants of the lines of the given family and the net angle ω; henceforth we will need only those terms of these expressions which contain derivatives of the function

$$\varkappa_2 = v^i\omega_i + \ldots, \qquad \sigma_{2,i} = [2\tau_1 \cos 2\omega - (\sigma_1 - \sigma_{1'}) \sin 2\omega]\, \omega_i + \ldots \tag{4.2}$$

where τ is the geodetic torsion and the ellipsis denotes terms not containing derivatives of ω.

We introduce the values of (4.2) into static vector formula (3.2) and then require that this vector be gradient by setting its curl equal to zero. The resulting expression constitutes a second-order differential equation in ω; the leading portion of this equation (i.e. the terms containing second derivatives) may be reduced to

$$\frac{1}{\sin^2\omega}\,[\sigma_2 u^i u^k - 2(\sigma_1 \cos\omega + \tau_1 \sin\omega)\, u^{(i}v^{k)} + \sigma_1 v^i v^k]\, \omega_{ik} + \ldots = h^{ik}\omega_{ik} + \ldots = 0 \tag{4.3}$$

where h^{ik} is the second tensor of the surface.

The discriminant of equation (4.3) is

$$\delta = \det |h^{ik}| = K / g \tag{4.4}$$

Since the quantity g (the discriminant of the metric tensor) is definitely positive, the sign of expression (4.4), and therefore the type of equation (4.3) are determined by the sign of the Gaussian curvature of the surface K. In addition, knowledge of the second surface tensor h^{ik} permits us immediately to specify the characteristics of equation (4.3) and the coordinate system in which it assumes the canonical form.

With the aid of the basic theorems of the general theory of differential equations, the result obtained can be formulated in the form of the following statement: any one-parameter family of lines can be combined with a second family with a degree of arbitrariness to within two functions of one argument in such a way that the resulting net is static.

5. We begin our consideration of special forms of static nets with the case of the orthogonal net ($\omega = \frac{1}{2}\pi$).

The rhombic vector of the orthogonal net A_k is equal to its Tchebyshev vector a_k ,

$$A_k = a_k = \varkappa_1 u_k + \varkappa_2 v_k \tag{5.1}$$

Hence, the static vector of the orthogonal net

$$\varphi_k = \frac{\sigma_{2,i}}{\sigma_2} u^i v_k - \frac{\sigma_{1,i}}{\sigma_1} v^i u_k + 2H\left(\frac{\varkappa_1}{\sigma_1} u_k + \frac{\varkappa_2}{\sigma_2} v_k\right) \tag{5.2}$$

($H = {}^1/_2(\sigma_1 + \sigma_2)$ is the average surface curvature)

The orthogonal static net will be sought as the result of rotating the net of curvature lines by an unknown angle ψ .

By means of operations completely similar to those carried out in Section 4, we arrive at a second-order equation in the function ψ; the leading portion of this equation is of the form

$$H\,[\sigma_2 u^i u^k - 2\tau_1 u^{(i} v^{k)} + \sigma_1 v^i v^k]\,\psi_{ik} + \ldots = Hh^{ik}\psi_{ik} + \ldots = 0 \tag{5.3}$$

Disregarding the case $H = 0$ for the moment, we may assume that orthogonal static nets exist on any surface and are determined with a degree of arbitrariness to within two functions of one argument.

As regards the case $H = 0$ (minimal surface), we have

$$\sigma_1 = -\sigma_2 = \sigma \tag{5.4}$$

and the static vector

$$\varphi_k = \frac{\sigma_i}{\sigma}(u^i v_k - u_k v^i) = -\partial_k \ln \sigma = \text{grad} \tag{5.5}$$

for an arbitrary orthogonal net.

Hence,

$$T = Ce^{\varphi} = C / \sigma \tag{5.6}$$

We thus arrive at the following theorem.

Theorem 5.1. An arbitrary orthogonal net of minimal surface is an instantaneously rigid net.

Another orthogonal net to consider is the net of curvature lines of an arbitrary surface. In this case, after a number of manipulations the static vector reduces to the form

$$\varphi_k = 2\left(\frac{\sigma_1}{\sigma_2}\varkappa_2 v_k + \frac{\sigma_2}{\sigma_1}\varkappa_1 u_k\right) = \partial_k \ln\frac{\sqrt{E}}{K} + \frac{H}{E}\rho_k{}^i\partial_i \ln\frac{K}{H} = 2\alpha_k{}^* \tag{5.7}$$

where $E = H^2 - K$ is the Euler difference, ρ_k^i is the fourth surface tensor, and $\alpha_k{}^*$ is the geodetic vector of the net of curvature lines relative to the asymptotic net.

Definition. A surface whose net of curvature lines is static is called a static surface.

Theorem 5.2. If the net of curvature lines of some surface is isothermal in the spherical representation, the surface is static; the static potential of its net of curvature lines is equal to twice the potential of the Tchebyshev vector of this set in the spherical representation,

$$\varphi = 2\rho \tag{5.8}$$

The validity of this statement follows directly from Norden's familiar theorem on conjugate connectivities [5] according to which the vector α_k^* is equal to the Tchebyshev vector ρ_k of the spherical representation of the net of curvature lines. Since the spherical representation of the net of curvature lines is an orthogonal net, the gradient of its Tchebyshev vector $\rho_k = \delta_k \rho$ implies that the net is isothermal.

It may be verified directly that static surfaces include, specifically, surfaces of rotation (the static potential is $\varphi = -\ln \sin^2\theta$, where θ is the angle between the normal to the surface and the axis of rotation), minimal surfaces (static potential: $\varphi = -\frac{1}{2}\ln|K|$), second-order surfaces ($\varphi = \frac{1}{2}(\frac{1}{2}\ln|K| - \ln E)$, and finally, surfaces with two families of plane curvature lines (whose spherical representations are the isothermal nets of Bonnet [5]). For $K < 0$ the nets of curvature lines of the foregoing surfaces are instantaneously rigid.

6. Let us now consider the arbitrary conjugate net of a surface. Taking into account the familiar relations among the lines of a conjugate net, the static vector of the latter may be expressed as

$$\begin{aligned}\varphi_k &= \frac{2}{\sin^2\omega}\left(\frac{\sigma_1}{\sigma_2}\varkappa_2 v_k + \frac{\sigma_2}{\sigma_1}\varkappa_1 u_k\right) + A_k - a_k + (2a_i - \alpha_i)\, c_k{}^i \cos\omega = \\ &= \left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right)\alpha_k - \left(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)\alpha_i a_k{}^i - \partial_k \ln \sin\omega\end{aligned} \tag{6.1}$$

where α_i is the geodetic vector of the net, a_k^i is its metrically normalized tensor, and $c_k{}^i$ is the metrically normalized tensor of the bisector net.

For the orthogonal conjugate net, i.e. for the set of curvature lines, expression (6.1) predictably becomes expression (5.7).

Vanishing of the geodetic vector, as we know, characterizes the geodetic net of the surface. Recalling that the normal curvatures of the conjugate directions are of opposite sign for $K < 0$, we infer from (6.1) the following theorem for a conjugate geodetic net (Foss net).

Theorem 6.1. A Foss net on a Foss surface of negative Gaussian curvature is an instantaneously rigid net; the static potential of the net is $\varphi = -\ln \sin\omega$.

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